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A LIAPUNOV FUNCTIONAL FOR A MATRIX NEUTRAL DIFFERENCE-DIFFERENTIAL EQUATION WITH ONE DELAY,

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A LIAPUNOV FUNCTIONAL FOR A MATRIX

NEUTRAL DIFFERENCE-DIFFERENTIAL EQUATION WITH ONE DELAY

Abstract: For the matrix neutral difference-differential equation $\dot{x}(t) + A\dot{x}(t-\tau) = Bx(t) + Cx(t-\tau)$ (dot) (dot) tau is constructed ~~we construct~~ a quadratic Liapunov functional which gives necessary and sufficient conditions for the asymptotic stability of the solutions of that equation. ~~We consider~~ a difference equation approximation is considered of the difference-differential equation, and for this difference equation ~~we construct~~ a Liapunov function is constructed from which ~~we~~ is obtained the desired Liapunov functional by an appropriate limiting process. The Liapunov functional thus obtained gives the best possible estimate for the rates of growth or decay of the solutions of the matrix neutral difference-differential equation. The results obtained are natural generalizations of previous results obtained for a matrix retarded difference-differential equation with one delay.

1. Introduction.

Consider the linear autonomous matrix neutral difference-differential equation with one delay

$$\dot{x}(t) + A\dot{x}(t-\tau) = Bx(t) + Cx(t-\tau), \quad t > 0, \tag{1.1}$$

where $x(t)$ is an n-vector function of time, A,B and C are constant $n \times n$ matrices and $\tau \geq 0$. Our purpose is to construct a Liapunov functional that characterizes the asymptotic behavior of the solutions of equation (1.1).

For a linear autonomous retarded difference-differential equation with one delay this same problem was considered recently in [9] for the scalar case, and in [8] for the matrix case. We will show that the procedures developed in [9] and [8] can be extended for the neutral difference-differential equation (1.1).

Several authors, notably Hale [3,6], Melvin [14] and Sendaula [15], have considered Liapunov functionals for neutral functional differential equations. Those Liapunov functionals only give sufficient conditions for the asymptotic stability of the solutions. The Liapunov functional constructed in this paper gives necessary and sufficient conditions for the asymptotic stability of the solutions of (1.1); moreover, it provides the best possible estimate for the rates of growth or decay of the solutions.

As in [9], we first consider a difference-equation approximation of the scalar equation

$$\dot{x}(t) + a\dot{x}(t-\tau) = bx(t) + cx(t-\tau), \quad t > 0. \tag{1.2}$$

For such a difference equation we construct, by means of well-known methods, a Liapunov function that gives necessary and sufficient conditions for the asymptotic stability of the solutions. Taking appropriate limits on this Liapunov function we obtain the desired Liapunov functional for equation (1.2).

The Liapunov functional for the matrix neutral difference-differential equation (1.1) is then obtained as a generalization of the one for the scalar equation. As in [8], the structure of the Liapunov functional is completely analyzed.

The Liapunov functional constructed depends critically on a matrix function which must satisfy a special functional differential equation. The existence, uniqueness and structure of the solutions of this equation are described. This special equation is a generalization of the one studied in [1].

2. The Neutral Difference-Differential Equation.

Let $L_2([a,b],\mathscr{R}^n)$ be the space of all Lebesgue square integrable functions defined on $[a,b]$ with values on $\mathscr{R}^n$, and denote by $W_2^1([a,b],\mathscr{R}^n)$ the space of all absolutely continuous functions which have the first derivative in $L_2([a,b],\mathscr{R}^n)$. With $\tau \geq 0$ fixed, consider the Hilbert space $\mathscr{A} = W_2^1([-\tau,0],\mathscr{R}^n)$ with the inner product

$$\langle\phi_1,\phi_2\rangle = \phi_1^T(0)\phi_2(0) + \int_{-\tau}^0 \phi_1^T(\theta)\phi_2(\theta)d\theta + \int_{-\tau}^0 \phi_1'^T(\theta)\phi_2'(\theta)d\theta$$

and the induced norm

$$||\phi||_{\mathscr{A}}^2 = \phi^T(0)\phi(0) + \int_{-\tau}^0 \phi^T(\theta)\phi(\theta)d\theta + \int_{-\tau}^0 \phi'^T(\theta)\phi'(\theta)d\theta.$$

Here, the superscript T denotes the transpose of a matrix. Let $x,\dot{x}$: $[-\tau,\infty) \to \mathscr{R}^n$; then for $t \geq 0$ we define the functions $x_t,\dot{x}_t$: $[-\tau,0] \to \mathscr{R}^n$ by $x_t(\theta) = x(t+\theta)$, $\dot{x}_t(\theta) = \dot{x}(t+\theta)$.

Consider the matrix neutral difference-differential equation

$$\dot{x}(t) + A\dot{x}(t-\tau) = Bx(t) + C(t-\tau), \qquad t > 0, \tag{2.1}$$

where A,B,C are $n \times n$ constant matrices, $x(t)$ is an n-vector and $\tau \geq 0$. Let

$$x_0 = \phi \tag{2.2}$$

be a given initial condition, with $\phi \in \mathcal{W}$.

A solution of the initial value problem (2.1)-(2.2) is a function $x\colon [-\tau,\infty) \to \mathbb{R}^n$ that satisfies the following conditions:

(a) $x_t \in \mathcal{W}$ for each $t \geq 0$;

(b) x satisfies the equation (2.1) a.e. (almost everywhere) on $[0,\infty)$; and

(c) $x_0 = \phi$.

It is known that the initial value problem (2.1)-(2.2) has a unique solution which depends continuously on the initial data in the norm of $\mathcal{W}$. A proof of this result is given by Melvin [13], who used the norm

$$||\phi|| = |\phi(-\tau)| + \left(\int_{-\tau}^{0} |\phi'^{T}(\theta)\phi'(\theta)|^2 d\theta\right)^{1/2},$$

which is easily seen to be equivalent to the norm defined on our Hilbert space.

We consider the solution operator $T(t)\colon \mathcal{W} \to \mathcal{W}$, defined by $T(t)\phi = x_t$, $\phi \in \mathscr{A}$, $t \geq 0$. The family $T(t)$, $t \geq 0$, is a C_0-semigroup. The infinitesimal generator $\mathscr{A}$ of $T(t)$ is given by $\mathscr{A}\phi = \phi'$ and it has a domain $\mathscr{D}(\mathscr{A})$, dense in $\mathscr{A}$, defined by

$$\mathscr{D}(\mathscr{A}) = \{\phi \in \mathscr{A} \mid \phi' \in \mathscr{A} \text{ and } \phi'(0) + A\phi'(-\tau) = B\phi(0) + C\phi(-\tau)\},$$

[3].

Let $\sigma(\mathscr{A})$ denote the spectrum of $\mathscr{A}$, i.e.,

$$\sigma(\mathscr{A}) = \{\lambda \mid \det[\lambda(I+Ae^{-\lambda\tau}) - B - Ce^{-\lambda\tau}] = 0\}. \tag{2.3}$$

Then, [3], there exists a constant γ such that $\operatorname{Re}(\lambda) \leq \gamma$ for all $\lambda \in \sigma(\mathscr{A})$. Also, for every $\varepsilon > 0$ there exists a constant $K \geq 1$ such that

$$||T(t)||_{(\mathscr{A},\mathscr{A})} \leq Ke^{(\gamma+\varepsilon)t}, \tag{2.4}$$

For our purposes, it is convenient to consider a representation of the solutions of (2.1) which is given for every $t,u \geq 0$ by the formula

$$\begin{aligned} x_{t+u}(0) = {} & [Y(t) + Y(t-\tau)A]x_t(0) + \\ & + \int_{-\tau}^{0} Y(u-\alpha-\tau)[Cx_t(\alpha) - Ax_t'(\alpha)]d\alpha, \end{aligned} \tag{2.5}$$

where the matrix Y is the solution of initial value problem

$$Y(t-u) = I + \int_u^{t-\tau} d_\beta Y(t-\beta-\tau)A + \int_u^t Y(t-\beta)Bd\beta +$$

$$+ \int_u^{t-\tau} Y(t-\beta-\tau)Cd\beta, \tag{2.6}$$

$$Y(0) = I,\ Y(t-u) = 0 \quad \text{for} \quad t < u.$$

The integrals in (2.6) are Lebesgue-Stieljes integrals, $Y(t-u)$ as a function of u is left continuous and $\operatorname{Var} Y(t-\cdot) < \infty$ on $[u,t]$ for every $t \geq u$, [7]. From (2.6) we have

$$Y(t-u) + Y(t-u-\tau)A = I + A +$$

$$+ \int_u^t [Y(t-\beta)B + Y(t-\beta-\tau)C]d\beta. \tag{2.7}$$

For $x_t \in \mathscr{D}(\mathscr{A})$ it follows from (2.5) that $Y(t-u) + Y(t-u-\tau)A$ is an absolutely continuous matrix valued function. Then from (2.7) we have that

$$\frac{d}{du}[Y(t-u) + Y(t-u-\tau)A] = -Y(t-u)B - Y(t-u-\tau)C, \text{ a.e.},$$

or

$$\frac{d}{dv}[Y(v) + Y(v-\tau)A] = Y(v)B + Y(v-\tau)C, \quad \text{a.e..} \tag{2.8}$$

3. A Liapunov Function for a Difference Equation Approximation of the Scalar Neutral Difference-Differential Equation.

Our purpose in this section is the construction of a Liapunov function for a difference equation approximation of the scalar neutral difference-differential equation

$$\dot{x}(t) + a\dot{x}(t-\tau) = bx(t) + cx(t-\tau), \qquad t \geq 0 . \tag{3.1}$$

Let N be fixed, and let the intervals $[0,\infty)$, $[-\tau,0]$ be subdivided in subintervals of equal length $\frac{\tau}{N}$. Denote the values of the function $x_t(\theta)$ at the mesh points $\left\{x_{k\frac{\tau}{N}}(-J\frac{\tau}{N})\right\}$, by x_k^J, $k = 0,1,\ldots$, $J = 0,\ldots,N$. We thus obtain the difference equation

$$\left\{\begin{array}{l} x_{k+1}^0 - x_k^0 + a(x_{k+1}^N - x_k^N) = \frac{\tau}{N}(bx_k^0 + cx_k^N) \\ \\ x_{k+1}^J = x_k^{J-1}, \quad J = 1,\ldots,N, \end{array}\right. \quad , \; k = 0,1,\ldots,$$

an approximation to (3.1). We assume, in this section, that $c \neq 0$, $a \neq 0$, and rewrite this difference equation in the form

$$y_{k+1} = \hat{A}y_k, \tag{3.2}$$

where y_k denotes the $(2N+1)$-dimensional vector

$$y_k = \left(cx_k^0, cx_k^1, \ldots, cx_k^N, -a\frac{x_k^0 - x_k^1}{\frac{\tau}{N}}, \ldots, -a\frac{x_k^{N-1} - x_k^N}{\frac{\tau}{N}}\right)^T, \tag{3.3}$$

and the $(2N+1) \times (2N+1)$ matrix $\hat{A}$ is given by

$$\hat{A} = \begin{bmatrix} (1+\frac{\tau}{N}b) & 0 & .\,. & 0 & \frac{\tau}{N}c & 0 & .\,.\,.\,. & 0 & \frac{\tau}{N}c \\ 1 & 0 & .\,. & 0 & 0 & 0 & .\,.\,.\,. & 0 & 0 \\ . & . & .\,. & . & . & . & .\,.\,.\,. & . & . \\ . & . & .\,. & . & . & . & .\,.\,.\,. & . & . \\ 0 & 0 & .\,. & 1 & 0 & 0 & & 0 & 0 \\ -\frac{ab}{c} & 0 & .\,. & 0 & -a & 0 & .\,.\,.\,. & 0 & -a \\ 0 & 0 & .\,. & 0 & 0 & 1 & .\,.\,.\,. & 0 & 0 \\ . & . & .\,. & . & . & . & .\,.\,.\,. & . & . \\ 0 & 0 & .\,. & 0 & 0 & 0 & .\,.\,.\,. & 1 & 0 \end{bmatrix} . \quad (3.4)$$

It is well-known how to construct a Liapunov function for the difference equation (3.2) so as to obtain necessary and sufficient conditions for the asymptotic stability of its solutions, [11,12]. For this purpose, we consider a Liapunov function given by $\hat{V}(y_k) = y_k^T D y_k$, where D is a positive definite matrix. Then the forward difference $\Delta\hat{V}(y_k) = \hat{V}(y_{k+1}) - \hat{V}(y_k)$ is given by $\Delta\hat{V}(y_k) = -y_k^T E y_k$, where $-E = \hat{A}^T D\hat{A} - D$. If E is a positive definite matrix, then the solutions will be asymptotically stable. On the other hand, if we have asymptotic stability, i.e., all eigenvalues of $\hat{A}$ have modulus strictly less than one, then for every positive definite matrix E there exists a unique positive definite matrix D that satisfies the equation $\hat{A}^T D\hat{A} - D = -E$. Moreover, if for some real number $\hat{\mu}$, $0 \leq \hat{\mu} < 1$, all the eigenvalues of the matrix $\frac{1}{\sqrt{1-\hat{\mu}}}\hat{A}$ have modulus less than one, then to every positive definite matrix E there corresponds a unique positive definite matrix D that satisfies the equation

$$\hat{A}D\hat{A} - (1-\hat{\mu})D = -E. \tag{3.5}$$

If $\hat{V}(y_k) = y_k^T D y_k$, then $\Delta\hat{V}(y_k) = -y_k^T E y_k - \hat{\mu} y_k^T D y_k \leq -\hat{\mu}\hat{V}(y_k)$.

The matrix $\hat{A}$ of particular interest here is seen to be equivalent to a matrix in companion form. In this case, for the existence, uniqueness and positive definiteness of the matrix D of equation (3.5) it suffices, [12], for the matrix E to be positive semidefinite and not identically equal to zero. This remark allows us to choose particularly simple matrices E; our purpose is to obtain as simple a form as possible for this matrix D and for this it is convenient, given the special form of the matrix $\hat{A}$, to restrict ourselves to certain choices of E. We shall represent the unique solution D of equation (3.5) in the form

$$D = \begin{bmatrix} \alpha & r^T & \tilde{r}^T \\ r & Q & \tilde{Q} \\ \tilde{r} & \tilde{Q} & \tilde{\tilde{Q}} \end{bmatrix} \tag{3.6}$$

where α is a scalar, $r^T = (r_1,\dots,r_N)$ and $\tilde{r}^T = (\tilde{r}_1,\dots,\tilde{r}_N)$ are N-dimensional vectors, $Q = (q_{ij})$ and $\tilde{\tilde{Q}} = (\tilde{\tilde{q}}_{ij})$ are $N \times N$ symmetric matrices and $\tilde{Q} = (\tilde{q}_{ij})$ is an $N \times N$ matrix.

The substitution of (3.4) and (3.6) into equation (3.5) suggests that a particularly simple form for the matrix D can be obtained if the matrix $E = E^T = (e_{ij})$ is chosen to have zero entries everywhere except the elements $e_{11}, e_{1,N+1} = e_{N+1,1}$, $e_{1,2N+1} = e_{2N+1,1}$, $e_{N+1,N+1}$, $e_{N+1,2N+1}$, $e_{2N+1,N+1}$ and $e_{2N+1,2N+1}$. Moreover, the simplicity of the structure of the matrix $\hat{A}$ allows for the vector $r^T = \tilde{r}^T = (r_1, \ldots, r_N)$ and for the matrices $Q, \tilde{Q}$ and $\tilde{\tilde{Q}}$ to be related by $q_{i,j} = \tilde{q}_{i,j} = \tilde{\tilde{q}}_{i,j}$ for $i \neq j$, $i,j = 1,\ldots,N$ and for $\tilde{q}_{ii} = 0$ for $i = 1,\ldots,N$ if the following equations are satisfied

$$q_{i,j} = (1-\hat{\mu})q_{i-1,j-1}, \quad i, j = 2,\ldots,N, \tag{3.7a}$$

$$\tilde{\tilde{q}}_{i,i} = (1-\hat{\mu})\tilde{\tilde{q}}_{i-1,i-1}, \quad i = 2,\ldots,N, \tag{3.7b}$$

$$r_i \frac{\tau}{N} c - aq_{i,1} - (1-\hat{\mu})q_{i-1,N} = 0, \quad i = 2,\ldots,N, \tag{3.7c}$$

$$(1 + \frac{\tau}{N} b)r_i + (1 - \frac{ab}{c})q_{1,i} - (1-\hat{\mu})r_{i-1} = 0, \quad i = 2,\ldots,N, \tag{3.7d}$$

and the nonzero elements of E are selected as

$$e_{1,N+1} = e_{N+1,1} = e_{1,2N+1} = e_{2N+1,1} =$$

$$= -[(\frac{\tau}{N} c\alpha - ar_1)(1 + \frac{\tau}{N} b) + (1 - \frac{ab}{c})\frac{\tau}{N} cr_1 + + \frac{a^2 b}{c} \tilde{\tilde{q}}_{11} - (1-\hat{\mu})r_N], \tag{3.7e}$$

$$e_{N+1,N+1} = -[(\frac{\tau}{N}c)^2\alpha - 2ar_1\frac{\tau}{N}c + a^2\tilde{\tilde{q}}_{11} - (1-\hat{\mu})q_{N,N}], \quad (3.7f)$$

$$e_{2N+1,2N+1} = -[(\frac{\tau}{N}c)^2\alpha - 2ar_1\frac{\tau}{N}c + a^2\tilde{\tilde{q}}_{11} - (1-\hat{\mu})\tilde{\tilde{q}}_{N,N}], \quad (3.7g)$$

$$e_{2N+1,N+1} = e_{N+1,2N+1} = -[(\frac{\tau}{N}c)^2\alpha - 2ar_1\frac{\tau}{N}c + a^2\tilde{\tilde{q}}_{11}]. \quad (3.7h)$$

From equations (3.7a-b) it follows that

$$q_{i,j} = (1-\hat{\mu})^{-N+i}q_{N,N-(i-j)}, \quad i > j \quad (3.8a)$$

$$q_{i,i} = (1-\hat{\mu})^{i-1}q_{1,1}, \quad i = 1,\ldots,N \quad (3.8b)$$

$$\tilde{\tilde{q}}_{i,i} = (1-\hat{\mu})^{i-1}\tilde{\tilde{q}}_{1,1}, \quad i = 1,\ldots,N \quad (3.8c)$$

Equations (3.7c-d), through use of equations (3.8a-c), yield

$$q_{N,i-2} - q_{N,i-1} + a(1-\hat{\mu})^{-N+i-1}(q_{N,N-(i-2)} - q_{N,N-(i-1)}) =$$

$$= \frac{\tau}{N}bq_{N,i-1} + \hat{\mu}\, q_{N,i-2} + (1-\hat{\mu})^{-N+i-1}q_{N,N-(i-1)}\frac{\tau}{N}c,$$

$$i = 3,\ldots,N, \quad (3.9)$$

$$r_1\frac{\tau}{N}c = (1-\tilde{\mu})^{-N+1}(a + \frac{\tau}{N}c)q_{N,N-1} + (1 + \frac{\tau}{N}b)q_{N,1}. \quad (3.10)$$

Now, letting $\tilde{V}(x_k) = (\frac{\tau}{N})^2 \hat{V}(y_k)$, where $x_k = (x_k^0, x_k^1, \ldots, x_k^N)$, $\tilde{\alpha} = (\frac{\tau}{N} c)^2 \alpha$, $\tilde{\beta} = \frac{\tau}{N} c^2 q_{11}$, $\tilde{\gamma} = a^2 \frac{\tau}{N} \tilde{\tilde{q}}_{11}$, $\mu = \frac{1}{2} \frac{N}{\tau} \hat{\mu}$, and using (3.7a-d), (3.8a-c), (3.9) and (3.10), we obtain the desired Liapunov function in the form

$$
\begin{aligned}
\tilde{V}(x_k) = {} & \tilde{\alpha}(x_k^0)^2 + \\
& + 2x_k^0 \sum_{i=2}^{N} [a(1 - 2\tfrac{\tau}{N}\mu)^{-N+i} q_{N,N-(i-1)} + (1-\hat{\mu}) q_{N,N-1}] \cdot \\
& \quad \cdot (cx_k^i - a \frac{x_k^{i-1} - x_k^i}{\frac{\tau}{N}}) \frac{\tau}{N} + \\
& + 2x_k^0 \frac{\tau}{N} [(1 - 2\tfrac{\tau}{N}\mu)^{-N+1}(a + \tfrac{\tau}{N} c) q_{N,N-1} + (1 + \tfrac{\tau}{N} b) q_{N,1}] \cdot \\
& \quad \cdot (cx_k^1 - a \frac{x_k^0 - x_k^1}{\frac{\tau}{N}}) + \\
& + 2 \sum_{i=2}^{N} \sum_{j=1}^{i-1} (cx_k^i - a \frac{x_k^{i-1} - x_k^i}{\frac{\tau}{N}})(1 - 2\tfrac{\tau}{N}\mu)^{-N+i} q_{N,N-(i-j)} \cdot \\
& \quad \cdot (cx_k^j - a \frac{x_k^{j-1} - x_k^j}{\frac{\tau}{N}}) \frac{\tau}{N} \frac{\tau}{N} + \\
& + \sum_{i=1}^{N} (1 - 2\tfrac{\tau}{N})^{-N+i} \tilde{\beta} (x_k^i)^2 \frac{\tau}{N} + \\
& + \sum_{i=1}^{N} (1 - 2\tfrac{\tau}{N})^{-N+i} \tilde{\gamma} (\frac{x_k^{i-1} - x_k^i}{\frac{\tau}{N}})^2 \frac{\tau}{N} ,
\end{aligned}
\tag{3.11}
$$

and its forward difference, divided by $\frac{\tau}{N}$,

$$\frac{\Delta\tilde{V}(x_k)}{\frac{\tau}{N}} = [2\tilde{\alpha}(b+\mu) + \tilde{\alpha}b^2\frac{\tau}{N} + \tilde{\beta} + b^2\tilde{\gamma} +$$

$$+ 2(1+\frac{\tau}{N}b)(c-ab)[(1-2\frac{\tau}{N}\mu)^{-N+1}(a+\frac{\tau}{N}c)q_{N,N-1} +$$

$$+ (1+\frac{\tau}{N}b)q_{N,1}](x_k^0)^2 +$$

$$+ [\tilde{\alpha}c^2\frac{\tau}{N} - 2\frac{\tau}{N}ac^2[(1-2\frac{\tau}{N}\mu)^{-N+1}(a+\frac{\tau}{N}c)q_{N,N-1} +$$

$$+ (1+\frac{\tau}{N}b)q_{N,1}] + \tilde{\gamma}c^2 - (1-2\frac{\tau}{N}\mu)^N\tilde{\beta}](x_k^N)^2 +$$

$$+ [\tilde{\alpha}a^2\frac{\tau}{N} - 2\frac{\tau}{N}a^3[(1-2\frac{\tau}{N}\mu)^{-N+1}(a+\frac{\tau}{N}c)q_{N,N-1} +$$

$$+ (1+\frac{\tau}{N}b)q_{N,1}] + \tilde{\gamma}a^2 - (1-2\frac{\tau}{N}\mu)^N\tilde{\gamma}]\left(\frac{x_k^{N-1} - x_k^N}{\frac{\tau}{N}}\right)^2 +$$

$$+ 2[\tilde{\alpha}c(1+\frac{\tau}{N}b) + [(1-2\frac{\tau}{N}\mu)^{-N+1}(a+\frac{\tau}{N}c)q_{N,N-1} +$$

$$+ (1+\frac{\tau}{N}b)q_{N,1}][(1-\frac{ab}{c})\frac{\tau}{N}c^2 - ac(1+\frac{\tau}{N}b)] +$$

$$+ bc\tilde{\gamma} - (1-2\frac{\tau}{N}\mu)[aq_{N,1} + (1-2\frac{\tau}{N}\mu)]q_{N,N-1}]c]x_k^0x_k^N +$$

$$+ 2[-\tilde{\alpha}a(1+\frac{\tau}{N}b) - [(1-2\frac{\tau}{N}\mu)^{-N+1}(a+\frac{\tau}{N}c)q_{N,N-1} +$$

$$+ (1+\frac{\tau}{N}b)q_{N,1}][(1-\frac{ab}{c})\frac{\tau}{N}ac - a^2(1+\frac{\tau}{N}b)] -$$

$$- ab\tilde{\gamma} + (1 - 2\frac{\tau}{N}\mu)[aq_{N,1} + (1 - 2\frac{\tau}{N}\mu)q_{N,N-1}]a]x_k^0 \frac{x_k^{N-1} - x_k^N}{\frac{\tau}{N}} +$$

$$+ 2[-\tilde{\alpha}ac\frac{\tau}{N} + 2a^2c\frac{\tau}{N}[(1 - 2\frac{\tau}{N}\mu)^{-N+1}(a + \frac{\tau}{N}c)q_{N,N-1} +$$

$$+ (1 + \frac{\tau}{N}b)q_{N,1}] - ca\tilde{\gamma}]x_k^N \frac{x_k^{N-1} - x_k^N}{\frac{\tau}{N}} - 2\mu\tilde{V}(x_k), \qquad (3.12)$$

where the off diagonal terms of the Q matrix must be related by

$$\frac{q_{N,i-2} - q_{N,i-1}}{\frac{\tau}{N}} - a(1 - 2\frac{\tau}{N}\mu)^{-N+i-1}\frac{q_{N,N-(i-1)} - q_{N,N-(i-2)}}{\frac{\tau}{N}} =$$

$$= bq_{N,i-1} + 2\mu q_{N,i-2} + (1 - 2\frac{\tau}{N}\mu)^{-N+i-1}cq_{N,N-(i-1)},$$

$$i = 3,\dots,N, \qquad (3.13)$$

4. A Liapunov Functional for the Scalar Neutral Difference-Differential Equation.

The results of Section 3 permit us to obtain a Liapunov functional in an explicit form for the scalar neutral difference-differential equation (2.1). For this purpose consider the limiting process described by

$$x_k(J) \xrightarrow[N\to\infty]{} x_t(\theta), \quad -\tau \le \theta \le 0,$$

$$q_{N,i} \xrightarrow[N\to\infty]{} q(-\theta), \quad -\tau \le \theta \le 0,$$

$$\tilde{V}(x_k) \xrightarrow[N\to\infty]{} V(x_t),$$

$$\frac{\Delta\tilde{V}(x_k)}{\frac{\tau}{N}} \xrightarrow[N\to\infty]{} \dot{V}(x_t),$$

$$\lim_{N\to\infty} (1 - 2\,\frac{\tau}{N}\,\mu)^{i-N+1} \to e^{2\mu\theta}, \qquad -\tau \leq \theta \leq 0.$$

These formal limits applied to equations (3.11), (3.12) and (3.13) yield the equations

$$\begin{aligned}
V(x_t) = {} & \tilde{\alpha} x_t^2(0) + \\
& + 2x_t(0)\int_{-\tau}^{0} (ae^{2\mu(\theta+\tau)}q(\tau+\theta) + q(-\theta))(cx_t(\theta) - ax_t'(\theta))d\theta + \\
& + 2\int_{-\tau}^{0}\int_{\theta}^{0} (cx_t(\theta) - ax_t'(\theta))e^{2\mu(\theta+\tau)}q(\tau+\theta-\beta)\cdot \\
& \qquad \cdot(cx_t(\beta) - ax_t'(\beta))d\beta d\theta + \\
& + \int_{-\tau}^{0} e^{2\mu\theta}\tilde{\beta}x_t^2(\theta)d\theta + \int_{-\tau}^{0} e^{2\mu\theta}\tilde{\gamma}\; x_t'^2(\theta)d\theta, \qquad (4.1)
\end{aligned}$$

$$\begin{aligned}
\dot{V}(x_t) = {} & [2\tilde{\alpha}(b+\mu) + \tilde{\beta} + b^2\tilde{\gamma} + \\
& \qquad + 2(c-ab)[aq(\tau)e^{2\mu\tau} + q(0)]]x_t^2(0) + \\
& + [\tilde{\gamma}c^2 - \tilde{\beta}e^{-2\mu\tau}]x_t^2(-\tau) + [\tilde{\gamma}a^2 - \tilde{\gamma}e^{-2\mu\tau}]\dot{x}_t^2(-\tau) + \\
& + 2c[\tilde{\alpha}-q(\tau) - 2aq(0) - a^2q(\tau)e^{2\mu\tau} + b\tilde{\gamma}]x_t(0)x_t(-\tau) -
\end{aligned}$$

$$- 2a[\tilde{\alpha} - q(\tau) - 2aq(0) - a^2 q(\tau)e^{2\mu\tau} + b\tilde{\gamma}]x_t(0)\dot{x}_t(-\tau) -$$

$$- 2ac\tilde{\gamma}x_t(-\tau)\dot{x}_t(-\tau) - 2\mu V(x_t), \qquad (4.2)$$

and

$$q'(-\theta) - ae^{2\mu(\theta+\tau)}q'(\theta+\tau) = -(b+2\mu)q(-\theta) -$$

$$-e^{2\mu(\theta+\tau)}cq(\theta+\tau). \qquad (4.3)$$

At this juncture, it is convenient to introduce the notation defined by the equations

$$q(\theta) = p(\tau-\theta)e^{\mu(\tau-\theta)},$$

and to let

$$\tilde{\alpha} = \tilde{\tilde{\alpha}} + p(0) + 2ae^{\mu\tau}p(\tau) + a^2 p(0)e^{2\mu\tau},$$
$$\tilde{\beta} = \tilde{\tilde{\beta}}e^{\mu\tau},$$
$$\tilde{\gamma} = \tilde{\tilde{\gamma}}e^{\mu\tau},$$

which introduced into (4.1), (4.2) and (4.3) yields the equations

$$V(x_t) = \tilde{\tilde{\alpha}}x_t^2(0) + e^{\mu\tau}\int_{-\tau}^{0} e^{2\mu\theta}\tilde{\tilde{\beta}}x_t^2(\theta)d\theta$$

$$+ e^{\mu\tau}\int_{-\tau}^{0} e^{2\mu\theta}\tilde{\tilde{\gamma}}x_t'^2(\theta)d\theta +$$

$$+ \left(p(0) + 2ae^{\mu\tau}p(\tau) + a^2p(0)e^{2\mu\tau}\right)x_t^2(0) +$$

$$+ 2x_t(0)\int_{-\tau}^{0} e^{\mu(\tau+\theta)}\left(p(\tau+\theta) + ap(-\theta)e^{\mu\tau}\right)\cdot$$

$$\cdot(cx_t(\theta) - ax_t'(\theta))d\theta +$$

$$+ 2\int_{-\tau}^{0}\int_{\theta}^{0}(cx_t(\theta) - ax_t'(\theta))e^{\mu(2\tau+\theta+\beta)}p(\beta-\theta)$$

$$\cdot(cx_t(\theta) - ax_t'(\theta))d\beta d\theta, \qquad (4.4)$$

$$\dot{V}(x_t) = [2(b+\mu)\tilde{\tilde{\alpha}} + e^{\delta\tau}(\tilde{\tilde{\beta}}+b^2\tilde{\tilde{\gamma}}) +$$

$$+ 2(b+\mu)p(0) + 2e^{\mu\tau}ap(\tau)(b+\mu) +$$

$$+ 2e^{2\mu\tau}a(c+a\mu)p(0) + 2e^{\mu\tau}(c+a\mu)p(\tau)]x_t^2(0) +$$

$$+ [\tilde{\tilde{\gamma}}c^2e^{\delta\tau} - \tilde{\tilde{\beta}}e^{-\delta\tau}]x_t^2(-\tau) + [e^{\delta\tau}a^2\tilde{\tilde{\gamma}} - e^{-\delta\tau}\tilde{\tilde{\gamma}}]\dot{x}_t^2(-\tau) +$$

$$+ 2c[\tilde{\tilde{\alpha}}+b\tilde{\tilde{\gamma}}e^{\delta\tau}]x_t(0)x_t(-\tau) -$$

$$- 2a[\tilde{\tilde{\alpha}}+b\tilde{\tilde{\gamma}}e^{\delta\tau}]x_t(0)\dot{x}_t(-\tau)$$

$$- 2ace^{\delta\tau}\tilde{\tilde{\gamma}}x_t(-\tau)\dot{x}_t(-\tau) -$$

$$- 2\mu V(x_t) \qquad (4.5)$$

and

$$p'(\theta) - ae^{\mu\tau}p'(\tau-\theta) = (b+\mu)p(\theta) +$$

$$+ (c+a\mu)e^{\mu\tau}p(\tau-\theta). \tag{4.6}$$

Equations (4.4), (4.5) and (4.6) have been obtained by formally taking limits on the Liapunov function for the discrete approximation to our original functional differential equation. It is easy to see that (4.4) is a well-defined functional on $\mathscr{A}$ and a straightforward although laborious computation shows that, if (4.6) is satisfied, (4.5) represents the rate of change of the functional (4.4) along the solutions of the scalar neutral functional differential equation with initial conditions in the domain of the generator; we postpone such an analysis, and use the results thus far obtained as motivation for the method of analysis presented in the next section.

5. A Liapunov Functional for the Matrix Neutral Difference-Differential Equation.

The results of the previous section suggest a form for a Liapunov functional for the matrix neutral difference-differential equation (2.1)-(2.2). For this purpose, on the space $\mathscr{A}$ consider the real symmetric quadratic form

$$V(\phi) = \phi^T(0)M\phi(0) + e^{\mu\tau}\int_{-\tau}^{0}\phi^T(\theta)Re^{2\mu\theta}\phi(\theta)d\theta +$$

$$+ e^{\mu\tau}\int_{-\tau}^{0}\phi^{T'}(\theta)Se^{2\mu\theta}\phi'(\theta)d\theta +$$

$$+ \varphi^T(0)[P(0) + e^{\mu\tau}A^TP^T(\tau) + e^{\mu\tau}P(\tau)A + e^{2\mu\tau}A^TP(0)A]\phi(0) +$$

$$+ 2\phi^T(0)\int_{-\tau}^{0} e^{\mu(\tau+\alpha)}(P(\alpha+\tau) + e^{\delta\tau}A^TP^T(-\alpha))\cdot(C\phi(\alpha) - A\phi'(\alpha))d\alpha +$$

$$+ 2\int_{-\tau}^{0}\int_{\alpha}^{0}(\phi^T(\alpha)C^T - \phi^{T'}(\alpha)A^T)P(\beta-\alpha)e^{\mu(2\tau+\alpha+\beta)}\cdot$$

$$\cdot(C\phi(\beta) - A\phi'(\beta))d\beta d\alpha, \tag{5.1}$$

where μ is a real number, M,R,S are constant $n \times n$ real positive definite matrices, and $P(\alpha)$, $0 \leq \alpha \leq \tau$, is a continuously differentiable matrix; we assume that $P(\alpha)$ is a solution of the initial value problem for the functional differential equation

$$P'(\alpha) - e^{\mu\tau}A^TP^{T'}(\tau-\alpha) = (B+\mu I)P(\alpha) +$$

$$+ e^{\mu\tau}(C^T + \mu A^T)P^T(\tau-\alpha), \quad 0 \leq \alpha \leq \tau, \tag{5.2}$$

$$P(0) = P^T(0) = P_0, \tag{5.3}$$

where P_0 is an arbitrary symmetric matrix.

Consider the Fréchet differentiable functional (5.1) evaluated along a solution of (2.1)-(2.2) with initial conditions in $\mathcal{D}(\mathcal{A})$; this yields a function of time, denoted by $V(t) = V(x_t)$, which is differentiable along such solution; after a straightforward but lengthy computation, and through use of (5.2)-(5.3), we obtain that

$$\dot{V}(t) = \frac{d}{dt} V(x_t) = \dot{V}(x_t) = -2\mu V(x_t) +$$

$$+ x_t^T(0)[(B^T+\mu I)M + M(B+\mu I) + 2e^{\mu\zeta}(R+B^TSB) +$$

$$+ (B^T+\mu I)P(0) + P(0)(B+\mu I) + e^{\mu\tau}(C^T+\mu A^T)P^T(\tau) +$$

$$+ e^{\mu\tau}P(\tau)(C+\mu A) + e^{\mu\tau}A^TP^T(\tau)(B+\mu I) +$$

$$+ e^{\mu\tau}(B^T+\mu I)P(\tau)A + e^{2\mu\tau}A^TP(0)(C+\mu A) +$$

$$+ e^{2\mu\tau}(C^T+\mu A^T)P(0)A]x_t(0) -$$

$$- e^{\mu\tau}h(x_t(0),x_t(-\tau),\dot{x}_t(-\tau)) \equiv U(x_t), \qquad (5.4)$$

where

$$h(x_t(0),\ x_t(-\tau),\ \dot{x}_t(-\tau)) = [x_t^T(0),\ -e^{-\mu\tau}x_t^T(-\tau),\ -e^{-\mu\tau}\dot{x}_t^T(-\tau)]\cdot$$

$$\cdot \begin{bmatrix} R + B^TSB & (M+e^{\mu\tau}B^TS)C & -(M+e^{\mu\tau}B^TS)A \\ C^T(M+e^{\mu\tau}SB) & R - e^{2\mu\tau}C^TSC & e^{2\mu\tau}C^TSA \\ -A^T(M+e^{\mu\tau}SB) & e^{2\mu\tau}A^TSC & S - e^{2\mu\tau}A^TSA \end{bmatrix} \begin{bmatrix} x_t(0) \\ -e^{-\mu\tau}x_t(-\tau) \\ -e^{-\mu\tau}\dot{x}_t(-\tau) \end{bmatrix},$$

$$(5.5).$$

This computation is valid only on solutions with initial data on $\mathscr{D}(\mathscr{A})$; however a direct application of Theorem 3.9 of [16] yields that for any solution with initial condition on $\mathscr{A}$ we have that for $\dot{V}(x_t) \equiv \overline{\lim_{\Delta\searrow 0}} \frac{V(x_{t+\Delta}) - V(x_t)}{\Delta} \leq U(x_t)$.

The objective of this paper is to show that it is possible to give an estimate of the rate of growth or decay of the solutions of the matrix neutral difference-differential equation (2.1)-(2.2), using a functional of the form (5.1) and its derivative (5.4).

For this purpose, let $\gamma = \max\{\mathrm{Re}|\lambda \in \sigma(\mathscr{A})\}$. Given $\varepsilon > 0$ and $-\mu = \gamma + 2\varepsilon$ we wish to show there exist matrices M,R,S and a differentiable matrix $P(\alpha)$ satisfying (5.2)-(5.3) so that for the functionals $V(\phi)$ and $U(\phi)$ given by (5.1) and (5.4) with these matrices we have that

$$c_1||\phi||^2_{\mathscr{A}} \leq V(\phi) \leq c_2||\phi||^2_{\mathscr{A}} \tag{5.6}$$

and

$$\dot{V}(\phi) \leq -2\mu V(\phi), \tag{5.7}$$

for some positive constants c_1, c_2. If this is possible, a norm is induced by the square root of the Liapunov functional (5.1), which we denote by $\tilde{||}\phi\tilde{||}_{\mathscr{A}} = \{V(\phi)\}^{1/2}$. Relationships (5.6) and (5.7) show that the norms $||\cdot||_{\mathscr{A}}$ and $\tilde{||}\cdot\tilde{||}_{\mathscr{A}}$ are equivalent on $\mathscr{A}$, and that

$$\tilde{||}x_t\tilde{||}_{\mathscr{A}} \leq \tilde{||}x_0\tilde{||}_{\mathscr{A}} e^{-\mu t}, \tag{5.8}$$

or

$$||x_t||_{\mathscr{C}} \leq \left[\frac{c_2}{c_1}\right]^{1/2} ||x_0||_{\mathscr{C}}\, e^{-\mu t}. \tag{5.9}$$

The estimates (5.8) and (5.9) are precisely those stated in (2.4). Note that the norm $\tilde{||}\cdot\tilde{||}_{\mathscr{C}}$ is the best possible one in the sense that it yields (2.4) with $K = 1$. Also, if $\gamma < 0$, it follows from (5.8) or (5.9) that the Liapunov functional (5.1) proves uniform exponential asymptotic stability for the solutions of (2.1).

In this manner, we have the following.

Theorem 1: Consider the matrix neutral difference-differential equation with one delay

$$\dot{x}(t) + A\dot{x}(t-\tau) = Bx(t) + Cx(t-\tau)$$

and the Liapunov functional given by equation (5.1). Let

$$\gamma = \max\{\mathrm{Re}\,\lambda \,|\, \det[\lambda(I+Ae^{-\lambda\tau}) - B - Ce^{-\lambda\tau}] = 0\}$$

and $\varepsilon > 0$. If there exist constant positive definite matrices M, R, S and a differentiable matrix $P(\alpha)$, $0 \leq \alpha \leq \tau$, that satisfies (5.2) with $P(0) = P(0)^T$, such that the inequalities (5.6) and (5.7) hold with c_1, c_2 positive and with $-\mu = \gamma + 2\varepsilon$; then the solutions of the difference-differential equation satisfy the exponential bound (5.9) and the equation is exponentially asymptotically stable if $\mu > 0$.

In the next two sections we show that an appropriate differentiable matrix $P(\alpha)$ as required by the above theorem can always be chosen, and we analyze its structure. In a further section we determine appropriate matrices M, R and S so that the conditions for the above theorem are always satisfied. We thus demonstrate the existence of a Liapunov functional of the stated form for the neutral functional differential equation.

At this juncture, we remark, for repeated later use, that for the particular above choice of $-\mu = \gamma + 2\varepsilon$, the spectral radius of $e^{\mu\tau}A$ is always less than one. Indeed, letting $x(t) = e^{-\mu t}y(t)$, $t \geq 0$, our original functional differential equation (2.1) becomes

$$\dot{y}(t) + e^{\mu\tau}A\dot{y}(t-\tau) = (B+\mu I)y(t) + (C+\mu A)e^{\mu\tau}y(t-\tau), \quad t \geq 0,$$

for which the solution operator is given by $T(t)e^{\mu t}$, where $T(t)$ is the solution operator of equation (2.1). Now, using equation (2.4) and $-\mu = \gamma + 2\varepsilon$, we note that $||T(t)e^{\mu t}||_{(\mathscr{A},\mathscr{A})} \leq Ke^{-\varepsilon t}$, $\varepsilon > 0$, which implies the uniform asymptotic stability of the solutions of the above equation. But, from [5], this implies that the solution to the difference equation $\tilde{y}(t) + e^{\mu\tau}A\tilde{y}(t-\tau) = 0$ is also uniformly asymptotically stable. But this can be the case if and only if the spectral radius of $e^{\mu\tau}A$ is strictly less than one.

6. The Functional Differential Equation for $P(\alpha)$.

In this section we consider the functional differential equation

$$P'(\alpha) - e^{\mu\tau}A^TP'^T(\tau-\alpha) = (B^T+\mu I)P(\alpha) +$$
$$+ (C^T+\mu A^T)P^T(\tau-\alpha), \quad -\infty < \alpha < \infty, \tag{6.1}$$

with the initial condition

$$P(\tfrac{\tau}{2}) = K, \tag{6.2}$$

where K is an arbitrary $n \times n$ matrix. Without any loss of generality, let $\mu = 0$ in equation (6.1), thus obtaining

$$P'(\alpha) - A^TP'^T(\tau-\alpha) = B^TP(\alpha) +$$
$$+ C^TP^T(\tau-\alpha), \quad -\infty < \alpha < \infty. \tag{6.1'}$$

For the particular case $A = 0$, namely

$$P'(\alpha) = B^TP(\alpha) + C^TP^T(\tau-\alpha), \quad -\infty < \alpha < \infty, \tag{6.1''}$$

it has been shown in [1] that the solutions of such an equation with initial conditions given by (6.2) exist and are unique and that the linear vector space of all solutions of such an equation has dimension n^2; moreover an algebraic representation of these solutions was presented. The results for the more general equation (6.1') are analogous to those presented in [1] for equation (6.1") and we only sketch them here. We assume in the sequel that all the eigenvalues of A lie strictly inside the unit circle.

Defining the matrix $F(\alpha) = P^T(\tau-\alpha)$, the functional differential equation (6.1') with the initial condition (6.2) reduces to the system of ordinary differential equations

$$\begin{aligned} P'(\alpha) + A^T F'(\alpha) &= B^T P(\alpha) + C^T F(\alpha) \\ P'(\alpha)A + F'(\alpha) &= -P(\alpha)C - F(\alpha)B \end{aligned} \tag{6.3}$$

or

$$\begin{aligned} P'(\alpha) - A^T P'(\alpha)A &= B^T P(\alpha) + A^T P(\alpha)C + C^T F(\alpha) + A^T F(\alpha)B \\ F'(\alpha) - A^T F'(\alpha)A &= -P(\alpha)C - B^T P(\alpha)A - F(\alpha)B - C^T F(\alpha)A \end{aligned} \tag{6.3'}$$

with initial conditions

$$P(\tfrac{\tau}{2}) = K, F(\tfrac{\tau}{2}) = K^T. \tag{6.4}$$

Consider the notation

$$P(\alpha) = (p_{ij}(\alpha)) = \begin{bmatrix} p_{1*}(\alpha) \\ \vdots \\ p_{n*}(\alpha) \end{bmatrix} = [p_{*1}(\alpha), \ldots, p_{*n}(\alpha)],$$

where $p_{i*}(\alpha)$ and $p_{*j}(\alpha)$ are, respectively, the i^{th} row and the j^{th} column of $P(\alpha)$. Then, equations (6.3') and (6.4) can be rewritten, through the use of the Kronecker (or direct) product of two matrices

[10] as the $2n^2$ system of ordinary differential equations

$$\begin{bmatrix} I - A \otimes A & 0 \\ 0 & I - A \otimes A \end{bmatrix} \frac{d}{d\alpha} \begin{bmatrix} p(\alpha) \\ f(\alpha) \end{bmatrix} =$$

$$= \begin{bmatrix} B \otimes I + A \otimes C & C \otimes I + A \otimes B \\ -I \otimes C - B \otimes A & -I \otimes B - C \otimes A \end{bmatrix} \begin{bmatrix} p(\alpha) \\ f(\alpha) \end{bmatrix}, \qquad (6.5)$$

with initial conditions

$$p(\tfrac{\tau}{\alpha}) = [k_{1*},\ldots,k_{n*}]^T,\ f(\alpha) = [k_{*1}^T,\ldots,k_{*n}^T]^T. \qquad (6.6)$$

Using the fact that 1 is not an eigenalue of A, then $I - A$ is invertible and so is $I - A \otimes A$. Thus the system (6.5) can be rewritten as

$$\frac{d}{d\alpha} \begin{bmatrix} p(\alpha) \\ f(\alpha) \end{bmatrix} =$$

$$\begin{bmatrix} (I - A \otimes A)^{-1} \cdot (B \otimes I + A \otimes C) & (I - A \otimes A)^{-1} \cdot (C \otimes I + A \otimes B) \\ (I - A \otimes A)^{-1} \cdot (-I \otimes C - B \otimes A) & (I - A \otimes A)^{-1} \cdot (-I \otimes B - C \otimes A) \end{bmatrix} \begin{bmatrix} p(\alpha) \\ f(\alpha) \end{bmatrix}, \qquad (6.5').$$

Consider now the uniqueness of solutions of (6.1')-(6.2). If this initial value problem has a differentiable solution $P(\alpha)$

defined on $-\infty < \alpha < \infty$, it follows that the pair of matrices $P(\alpha), F(\alpha)$ with $F(\alpha) = P^T(\tau-\alpha)$, will satisfy (6.3), (6.4) and then the pair of vectors $p(\alpha), f(\alpha)$ as defined above, will satisfy (6.5'), (6.6). Because of the linearity of all the equations, and the uniqueness of the solutions of (6.5'), (6.6), it follows that the solution $P(\alpha)$ of (6.1'), (6.2) is unique.

Consider now the question of existence of solutions of (6.1')-(6.2). The initial value problem (6.5')-(6.6) has a unique solution $(p(\alpha), f(\alpha))$ defined on $-\infty < \alpha < \infty$. This implies the existence of a unique pair of differentiable matrices $P(\alpha), F(\alpha)$ defined on $-\infty < \alpha < \infty$ and satisfying (6.3), (6.4). These equations can be rewritten as

$$\frac{d}{d\alpha} P(\alpha) + A^T \frac{d}{d\alpha} F(\alpha) = B^T P(\alpha) + C^T F(\alpha)$$

$$\frac{d}{d\alpha} F^T(\tau-\alpha) + A^T \frac{d}{d\alpha} P^T(\tau-\alpha) = B^T F(\tau-\alpha) + C^T P(\tau-\alpha),$$

with the initial condition

$$P(\frac{\tau}{2}) = K = F^T(\frac{\tau}{2}).$$

Then it follows, from the uniqueness of the solutions, that $F(\alpha) = P^T(\tau-\alpha)$ for $-\infty < \alpha < \infty$.

Moreover, it is well known, [2,4], that the system (6.5') has $2n^2$ linearly independent solutions which can be represented in the following fashion. Let $\lambda_1,\ldots,\lambda_\ell$, $\ell < 2n^2$, be the distinct eigenvalues of the matrix in (6.5'), that is, solutions of the determinental equation

$$\det\begin{bmatrix} (I - A\otimes A)\lambda - (B\otimes I + A\otimes C) & -C\otimes I - A\otimes B \\ I\otimes C + B\otimes A & (I - A\otimes A)\lambda + (I\otimes B \oplus C\otimes A) \end{bmatrix} = 0; \quad (6.7)$$

each λ_j, $j = 1,\ldots,\ell$, with algebraic multiplicity m_j and geometric multiplicities n_j^r, $\sum_{r=1}^{s} n_j^r = m_j$, $\sum_j m_j = 2n^2$. Then, the $2n^2$ linearly independent solutions of (6.5') are given by

$$\phi_{j,r}^{q}(\alpha) = e^{\lambda_j(\alpha - \frac{\tau}{2})} \sum_{i=1}^{q} \frac{(\alpha - \frac{\tau}{2})^{q-i}}{(q-i)!} e_{j,r}^{i}\,, \quad (6.8)$$

where $q = 1,\ldots,n_j^r$, $r = 1,\ldots,s$, $\sum_{r=1}^{s} n_j^r = m_j$, $\sum_j m_j = 2n^2$, and the $2n^2$ linearly independent eigenvectors and generalized eigenvectors are given by

$$[\lambda_j I - H]e_{j,r}^{i} = -e_{j,r}^{i-1}\,, \quad e_{j,s}^{0} = 0,$$

where H is the $2n^2 \times 2n^2$ matrix in (6.5'). Changing notation and returning from the vector to the matrix form, we see that the $2n^2$ linearly independent solutions of (6.3) are given by

$$\begin{bmatrix} \Phi_{j,r}^{q}(\alpha) \\ \Psi_{j,r}^{q}(\alpha) \end{bmatrix} = e^{\lambda_j(\alpha - \frac{\tau}{2})} \sum_{i=1}^{q} \left[\frac{(\alpha - \frac{\tau}{2})^{q-i}}{(q-i)!} \begin{bmatrix} L_{j,r}^{i} \\ M_{j,r}^{i} \end{bmatrix} \right], \quad (6.9)$$

for $q = 1,\ldots,n_j^r$, $r = 1,\ldots,s$, $j = 1,\ldots,p$, $\sum_{n=1}^{s} n_j^r = m_j$, $\sum m_j = 2n^2$, where the generalized eigenmatrix pair $(L_{j,r}^i, M_{j,r}^i)$ associated with the eigenvalue λ_j satisfies the equations

$$(\lambda_j I-B)L_{j,r}^i - AL_{j,r}^i(C^T+\lambda A^T) - CM_{j,r}^i - AM_{j,r}^i B^T = AL_{j,r}^{i-1}A^T - L_{j,r}^{i-1}\,,$$
$$L_{j,r}^i C^T + BL_{j,r}^i A^T + M_{j,r}^i(\lambda_j I+B^T) + (C-\lambda A)M_{j,r}^i A^T = AM_{j,r}^{i-1}A^T - M_{j,r}^{i-1} \qquad (6.10)$$

for $i = 1,\ldots,n_j^r$, $r = 1,\ldots,s$, $L_{j,s}^0 = 0$, $M_{j,s}^0 = 0$. The structure of these equations is very special; in fact, if they are multiplied by -1, transposed, and written in reverse order, we have

$$(-\lambda_j I-B)M_{j,r}^{i^T} - AM_{j,r}^{i^T}(C^T-\lambda A^T) - CL_{j,r}^{i^T} - AL_{j,r}^{i^T}B^T = -AM_{j,r}^{i-1^T}A^T + M_{j,r}^{i-1^T}\,,$$
$$M_{j,r}^{i^T}C^T + BM_{j,r}^{i^T}A^T + L_{j,r}^{i^T}(-\lambda_j I+B^T) + (C+\lambda A)L_{j,r}^{i^T}A^T = -AL_{j,r}^{i-1^T}A^T + L_{j,r}^{i-1^T}\,, \qquad (6.11)$$

for $i = 1,\ldots,n_j^r$, $r = 1,\ldots,s$, $L_{j,r}^{0^T} = M_{j,r}^{0^T} = 0$. This result shows that if λ_j is a solution of (6.7), $-\lambda_j$ will also be a solution; moreover, λ_j and $-\lambda_j$ have the same geometric multiplicity and the same algebraic multiplicity. Thus, the distinct eigenvalues always appear in pairs $(\lambda_j,-\lambda_j)$. An examination of equations (6.10) and (6.11) shows that if the generalized eigenmatrix pair corresponding to λ_j are $(L_{j,r}^i, M_{j,r}^i)$, then the generalized eigenmatrix pair corresponding to $-\lambda_j$ will be $((-1)^{i+1}M_{j,r}^{i^T}, (-1)^{i+1}L_{j,r}^{i^T})$. These remarks show that if the solution (6.8) corresponding to λ_j is added to the solution (6.8) corresponding to $-\lambda_j$ multiplied by $(-1)^{q+1}$, we have n^2 linearly independent solutions of (6.3) given by

$$\begin{bmatrix} \Xi^q_{j,s}(\alpha) \\ \pi^q_{j,s}(\alpha) \end{bmatrix} = e^{\lambda_j(\alpha - \frac{\tau}{2})} \sum_{i=1}^{q} \frac{(\alpha - \frac{\tau}{2})^{q-i}}{(q-i)!} \begin{bmatrix} L^i_{j,r} \\ M^i_{j,r} \end{bmatrix} +$$

$$+ e^{-\lambda_j(\alpha - \frac{\tau}{2})} \sum_{i=1}^{q} \frac{(\alpha - \frac{\tau}{2})^{q-i}}{(q-i)!} (-1)^{q+i} \begin{bmatrix} M^{i^T}_{j,r} \\ L^{i^T}_{j,r} \end{bmatrix},$$

which satisfy the condition $\Xi^q_{j,r}(\frac{\tau}{2}) = \pi^{q^T}_{j,r}(\frac{\tau}{2})$. This condition is precisely condition (6.2); it therefore follows that

$$\Xi^q_{j,r}(\alpha) = \sum_{i=1}^{q} \frac{(\alpha - \frac{\tau}{2})^{q-i}}{(q-i)!} \Bigg[e^{\lambda_j(\alpha - \frac{\tau}{2})} L^i_{j,r} + \tag{6.12}$$

$$+ (-1)^{q+i} e^{-\lambda_j(\alpha - \frac{\tau}{2})} M^{i^T}_{j,r} \Bigg],$$

for $q = 1,\ldots,n^r_j$, $\sum_{r=1}^{s} n^r_j = m_j$, $\sum_{2j} m_j = 2n^2$, are n^2 linearly independent solutions of (5.10'). Hence, we have shown that equation (6.1') has n^2 linearly independent solutions given by equation (6.12), where the generalized eigenmatrix pairs $(L^i_{j,s}, M^i_{j,s})$ satisfy equation (6.10) for one of the elements of the pair $(\lambda_j, -\lambda_j)$, each of which is a solution of equation (6.7).

Making use of the above results, it is clear that the functional differential equation (5.2) with initial condition (5.3) has a unique solution $P(\alpha)$ defined for $-\infty < \alpha < \infty$.

7. The Matrix Function $\tilde{P}(\alpha)$ and Its Properties.

In this section we show that the solution $P(\alpha)$ of (6.1) has an integral representation which is particularly useful for our purposes. To this end consider the matrix function $\tilde{P}(\alpha)$ defined by

$$\tilde{P}(\alpha) = \int_0^\infty Y^T(\beta)e^{\mu\beta}WY(\beta-\alpha)e^{\mu(\beta-\alpha)}d\beta, \quad \alpha \geq 0, \tag{7.1}$$

where W is an arbitrary symmetric matrix, $Y(\beta)$ is the solution of equation (2.6) and $\mu = -\gamma - 2\varepsilon$, $\varepsilon > 0$. This integral always converges since, for every $\varepsilon > 0$, we have $||Y(\beta)|| \leq \tilde{K}e^{(\gamma+\varepsilon)\beta}$ for some $\tilde{K} \geq 1$.

From (7.1) it immediately follows that

$$\tilde{P}(0) = \tilde{P}^T(0) = \int_0^\infty Y^T(\beta)e^{\mu\beta}WY(\beta)e^{\mu\beta}d\beta, \tag{7.2}$$

and that

$$\begin{aligned}\tilde{P}(\alpha) &= \int_\alpha^\infty Y^T(\beta)e^{\mu\beta}WY(\beta-\alpha)e^{\mu(\beta-\alpha)}d\beta = \int_0^\infty Y^T(\alpha+\xi)e^{\mu(\alpha+\xi)}WY(\xi)e^{\mu\xi}d\xi \\ &= P^T(-\alpha), \qquad -\infty < \alpha < \infty.\end{aligned} \tag{7.3}$$

We now show that $\tilde{P}(\alpha)$ satisfies the equation

$$\begin{aligned}\tilde{P}'(\alpha) - e^{\mu\tau}A^T\tilde{P}'^T(\tau-\alpha) &= (B^T+\mu I)\tilde{P}(\alpha) + \\ &+ (C^T+\mu A^T)e^{\mu\tau}\tilde{P}^T(\tau-\alpha), \quad 0 \leq \alpha \leq \tau.\end{aligned} \tag{7.4}$$

Indeed, using equation (2.6) in the definition of $\tilde{P}(\alpha)$, we have

$$\tilde{P}(\alpha) = \int_0^{\infty} Y^T(\beta)e^{\mu\beta}We^{\mu(\beta-\alpha)}(I+A)d\beta -$$

$$- \int_0^{\infty} Y^T(\beta)e^{\mu\beta}We^{\mu(\beta-\alpha)}Y(\beta-\alpha-\tau)Ad\beta +$$

$$+ \int_0^{\infty}\int_{\alpha}^{\beta} Y^T(\beta)e^{\mu\beta}We^{\mu(\beta-\alpha)}Y(\beta-\xi)Bd\xi d\beta +$$

$$+ \int_0^{\infty}\int_{\alpha}^{\beta-\tau} Y^T(\beta)e^{\mu\beta}We^{\mu(\beta-\alpha)}Y(\beta-\xi-\tau)Cd\xi d\beta.$$

The order of integration in the last two terms can be interchanged, yielding

$$P(\alpha) = e^{-\mu\alpha}\int_0^{\infty} Y^T(\beta)e^{\mu\beta}We^{\mu\beta}(I+A)d\beta -$$

$$- e^{\mu\tau}\int_0^{\infty} Y^T(\beta)e^{\mu\beta}We^{\mu(\beta-\alpha-\tau)}Y(\beta-\alpha-\tau)Ad\beta +$$

$$+ \int_{\alpha}^{\infty}\left[\int_{\xi}^{\infty} Y^T(\beta)e^{\mu\beta}We^{\mu(\beta-\xi)}Y(\beta-\xi)d\beta\right]Be^{\mu(\xi-\alpha)}d\xi +$$

$$+ \int_{\alpha}^{\infty}\left[\int_{\xi\ \tau}^{\infty} Y^T(\beta)e^{\mu\beta}We^{\mu(\beta-\xi-\tau)}Y(\beta-\xi-\tau)d\beta\right]Ce^{\mu(\xi-\alpha+\tau)}d\xi$$

or

$$\tilde{P}(\alpha) = e^{-\mu\alpha}\int_0^{\infty} Y^T(\beta)e^{\mu\beta}We^{\mu\beta}(I+A)d\beta - e^{\mu\tau}\tilde{P}(\alpha+\tau)A +$$

$$+ e^{-\mu\alpha}\int_{\alpha}^{\infty}[\tilde{P}(\xi)e^{\mu\xi}B + \tilde{P}(\xi+\tau)e^{\mu(\xi+\tau)}C]d\xi. \qquad (7.5)$$

Every term in (7.5) is an absolutely continuous function; hence differentiation of (7.5) gives

$$\frac{d}{d\alpha}\tilde{P}(\alpha) = -e^{\mu\tau}[\frac{d}{d\alpha}\tilde{P}(\alpha+\tau)]A - \tilde{P}(\alpha)B - e^{\mu\tau}\tilde{P}(\alpha+\tau)C -$$

$$- \mu(e^{-\mu\alpha}\int_0^\infty Y^T(\beta)e^{\mu\beta}We^{\mu\beta}(I+A)d\beta +$$

$$+ e^{-\mu\alpha}\int_\alpha^\infty [\tilde{P}(\xi)e^{\mu\xi}B + \tilde{P}(\xi+\tau)e^{\mu(\xi+\tau)}C]d\xi)$$

or

$$\frac{d}{d\alpha}\tilde{P}(\alpha) = -e^{\mu\tau}[\frac{d}{d\alpha}\tilde{P}(\alpha+\tau)]A - \tilde{P}(\alpha)B - e^{\mu\tau}\tilde{P}(\alpha+\tau)C -$$

$$- \mu(\tilde{P}(\alpha) + e^{\mu\tau}\tilde{P}(\alpha+\tau)A)$$

or

$$\frac{d}{d\alpha}\tilde{P}(\alpha) + e^{\mu\tau}[\frac{d}{d\alpha}\tilde{P}(\alpha+\tau)]A = -\tilde{P}(\alpha)(B+\mu I) -$$

$$- \tilde{P}(\alpha+\tau)(C+\mu A)e^{\mu\tau}$$

or

$$\tilde{P}'(\alpha) + e^{\mu\tau}\tilde{P}'(\alpha+\tau)A = -\tilde{P}(\alpha)(B+\mu I) - \tilde{P}(\alpha+\tau)(C+\mu A)e^{\mu\tau} .$$

We have, therefore, that $\tilde{P}(\alpha)$ satisfies the equation

$$\tilde{P}'(-\alpha) + e^{\mu\tau}\tilde{P}'(-\alpha+\tau)A = -\tilde{P}(-\alpha)(B+\mu I) - \tilde{P}(-\alpha+\tau)(C+\mu A)e^{\mu\tau};$$

but using property (7.3), we obtain that

$$\tilde{P}'(-\alpha) = -\frac{d}{d\alpha}\tilde{P}(-\alpha) = -\frac{d}{d\alpha}\tilde{P}^T(\alpha) = -\tilde{P}'^T(\alpha).$$

Hence

$$-\tilde{P}'^T(\alpha) + e^{\mu\tau}\tilde{P}'(-\alpha+\tau)A = -\tilde{P}^T(\alpha)(B+\mu I) - $$
$$- \tilde{P}(-\alpha+\tau)(C+\mu A)e^{\mu\tau}$$

or

$$\tilde{P}'(\alpha) - e^{\mu\tau}A^T\tilde{P}'^T(\tau-\alpha)A = (B^T+\mu I)\tilde{P}(\alpha) +$$
$$+ e^{\mu\tau}(C^T+\mu A^T)\tilde{P}^T(\tau-\alpha).$$

Because of the uniqueness of the solutions of (5.2)-(5.3), it is seen that $\tilde{P}(\alpha)$ defined by (7.1) is the unique solution of (5.2)-(5.3) with the initial condition prescribed by (7.2).

We now proceed to show that the following relationship is satisfied:

$$\begin{aligned} Z \equiv\ & (B^T+\mu I)\tilde{P}(0) + \tilde{P}(0)(B+\mu I) + \\ & + e^{\mu\tau}(C^T+\mu A^T)\tilde{P}^T(\tau) + e^{\mu\tau}\tilde{P}(\tau)(C+\mu A) + \\ & + e^{\mu\tau}A^T\tilde{P}^T(\tau)(B+\mu I) + e^{\mu\tau}(B^T+\mu I)\tilde{P}(\tau)A + \\ & + e^{2\mu\tau}A^T\tilde{P}(0)(C+\mu A) + e^{2\mu\tau}(C^T+\mu A^T)\tilde{P}(0)A = -W. \end{aligned} \tag{7.6}$$

Indeed, using $\tilde{P}(0)$ and $\tilde{P}(\tau)$ given by (7.1) and the relation

$$e^{2\mu\tau}\tilde{P}(0) = \int_0^\infty Y^T(\beta-\tau)e^{\mu\beta}WY(\beta-\tau)e^{\mu\beta}d\beta,$$

it is easy to see Z can be represented as

$$Z = \int_0^\infty [B^TY(\beta) + C^TY^T(\beta-\tau)]e^{\mu\beta}We^{\mu\beta}[Y(\beta) + Y(\beta-\tau)A]d\beta +$$

$$+ \int_0^\infty [Y^T(\beta) + A^TY^T(\beta-\tau)]e^{\mu\beta}We^{\mu\beta}[Y(\beta)B + Y(\beta-\tau)C]d\beta +$$

$$+ 2\mu \int_0^\infty [Y^T(\beta) + A^TY^T(\beta-\tau)]e^{\mu\beta}We^{\mu\beta}[Y(\beta) + Y(\beta-\tau)A]d\beta.$$

Making use of (2.3), we have that

$$Z = \int_0^\infty \frac{d}{d\beta} [[Y^T(\beta) + A^TY^T(\beta-\tau)]e^{\mu\beta}We^{\mu\beta}[Y(\beta) + Y(\beta-\tau)A]]d\beta,$$

from where it follows that $Z = -W$.

We remark that to each constant symmetric positive definite matrix W there corresponds a matrix function $\tilde{P}(\alpha)$ given by (7.1). This matrix $\tilde{P}(\alpha)$ is the unique solution of (5.2)-(5.3) with the initial condition prescribed by $\tilde{P}(0) = \tilde{P}(0)^T = \int_0^\infty Y^T(\beta)e^{\mu\beta}WY(\beta)e^{\mu\beta}d\beta$ which is necessarily positive definite. Conversely, to each $P(0) = P^T(0) = P_0$, equations (5.2)-(5.3) have a unique differentiable solution $P(\alpha)$ which, through (7.6) defines a unique symmetric matrix W with the property that for this W, (7.1) gives the unique solution for (5.2)-(5.3) with the prescribed initial condition. It follows that the map $W \to \tilde{P}(0)$, defined by

$$\tilde{P}(0) = \int_0^\infty Y^T(\beta)e^{\mu\beta}WY(\beta)e^{\mu\beta}d\beta,$$

as a map on the space of $n \times n$ symmetric matrices is one-to-one, onto, and it maps positive definite matrices W into positive definite matrices $\tilde{P}(0)$.

8. Structure of the Liapunov Functional and Its Derivative.

The characterization given in the previous sections for the matrix function $\tilde{P}(\alpha)$ permit us to put into evidence the particular structure of the Liapunov functional (5.1) and its derivative (5.4). In fact, the substitution of the representation (7.1) for $P(x)$ into the Liapunov functional (5.1), yields

$$\begin{aligned} V(\phi) = {} & \phi^T(0)M\phi(0) + e^{\mu\tau}\int_{-\tau}^0 \phi^T(\theta)Re^{2\mu\theta}\phi(\theta)d\theta + \\ & + e^{\mu\tau}\int_{-\tau}^0 \phi'^T(\theta)Se^{2\mu\theta}\phi'(\theta)d\theta + \\ & + \int_0^\infty \Big\{[Y(\beta) + Y(\beta-\tau)A]\phi(0) + \\ & + \int_{-\tau}^0 Y(\beta-\alpha-\tau)[C\phi(\alpha) - A\phi'(\alpha)]d\alpha\Big\}^T We^{2\mu\beta}\cdot \\ & \cdot\Big\{[Y(\beta) + Y(\beta-\tau)A]\phi(0) + \\ & + \int_{-\tau}^0 Y(\beta-\alpha-\tau)[C\phi(\alpha) - A\phi'(\alpha)]d\alpha\Big\}d\beta. \end{aligned}$$

But this functional, using (2.5), can be represented as

$$V(x_t) = x_t^T(0)Mx_t(0) + e^{\mu\tau}\int_{-\tau}^{0} x_t^T(\theta)Re^{2\mu\theta}x_t(\theta)d\theta$$

$$+ e^{\mu\tau}\int_{\tau}^{0} x_t'^T(\theta)Se^{2\mu\theta}x_t'(\theta)d\theta +$$

$$+ \int_0^{\infty} x_{t+\beta}(0)We^{2\mu\beta}x_{t+\beta}(0)d\beta, \tag{8.1}$$

where $x_{t+\beta}(0)$ is the solution of (2.1)-(2.2) for $\beta \geq 0$ with initial condition $x_t(\theta)$, $-\tau \leq \theta \leq 0$.

In an analogous manner, using the above notation, equation (5.4) can be rewritten as

$$\dot{V}(x_t) = -2\mu V(x_t) +$$

$$+ x_t^T(0)[-W + (A^T+\mu I)M + M(A+\mu I) + 2e^{\mu\tau}(R+B^TSB)]x_t(0)$$

$$- e^{\mu\tau}[x_t^T(0),\ - e^{\mu\tau}x_t^T(-\tau),\ \ -e^{-\mu\tau}\dot{x}_t^T(-\tau)]\cdot$$

$$\cdot \begin{bmatrix} R + B^TSB & [M+e^{\mu\tau}B^TS]C & -[M+e^{\mu\tau}B^TS]A \\ C^T[M+e^{\mu\tau}SB] & R - e^{2\mu\tau}C^TSC & e^{2\mu\tau}C^TSA \\ -A^T[M+e^{\mu\tau}SB] & e^{2\mu\tau}A^TSC & S - e^{2\mu\tau}A^TSA \end{bmatrix} \begin{bmatrix} x_t(0) \\ -e^{-\mu\tau}x_t(-\tau) \\ -e^{-\mu\tau}\dot{x}_t(-\tau) \end{bmatrix}. \tag{8.2}$$

9. Bounds for the Liapunov Functional and Its Derivative.

The structure of the Liapunov functional (5.1) and its derivative (5.4) developed in the previous sections allows us to establish that it is, always possible to choose positive definite matrices M,R,S and W (and therefore $P(\alpha)$) so that bounds of the form (5.6) and (5.7) always hold. For this purpose, first we will show that there exist positive constants c_1, c_2 such that

$$c_1 ||\phi||^2_{\mathscr{A}} \leq V(\phi) \leq c_2 ||\phi||^2_{\mathscr{A}} \tag{9.1}$$

for all $\phi \in \mathscr{D}(\mathscr{A})$.

Denote by $\lambda_{min}(M)$ and $\lambda_{max}(M)$ the minimum and maximum eigenvalues of the positive definite matrix M, the corresponding notation to be used for the positive definite matrices R,S and W.

From (8.1) it immediately follows that

$$\min(\lambda_{min}(M), e^{-|\mu|\tau}\lambda_{min}(R), e^{-|\mu|\tau}\lambda_{min}(S)) ||\phi||^2_{\mathscr{A}} \leq V(\phi).$$

Using (2.4), it is easily seen that

$$||x_\beta(0)||_{\mathscr{R}^n} \leq ||x_\beta||_{\mathscr{A}} \leq Ke^{(\gamma+\varepsilon)\beta}||x_0||_{\mathscr{A}} \quad ;$$

Thus, from (8.1) it follows that

$$V(\phi) \leq [\max(\lambda_{max}(M), e^{|\mu|\tau}\lambda_{max}(R), e^{|\mu|\tau}\lambda_{max}(S)) +$$

$$+ \frac{\lambda_{max}(W)}{2\varepsilon} K^2] ||\phi||^2_{\mathcal{C}} \quad .$$

The desired values of c_1, c_2 will therefore be given by

$$c_1 = \min(\lambda_{min}(M), e^{-|\mu|\tau}\lambda_{min}(R), e^{-|\mu|\tau}\lambda_{min}(S)),$$

and

$$c_2 = \max(\lambda_{min}(M), e^{|\mu|\tau}\lambda_{max}(R), e^{|\mu|\tau}\lambda_{max}(S)) + \frac{\lambda_{max}(W)}{2\varepsilon} K^2.$$

Secondly, we proceed to show that equation (5.7) always holds. Equation (8.2) shows that we only need to choose the positive definition matrices W, R, S and M in such a manner that the function

$$h_1(x_t(0), x_t(-\tau), \dot{x}_t(-\tau)) = [x_t^T(0), -e^{-\mu\tau}x_t^T(-\tau), -e^{-\mu\tau}\dot{x}_t^T(-\tau)] \cdot$$

$$\cdot \begin{bmatrix} W-(A^T+\mu I)M-M(A+\mu I) & [M+e^{\mu\tau}B^TS]Ce^{\mu\tau} & -[M+e^{\mu\tau}B^TS]Ae^{\mu\tau} \\ -(R+B^TSB)e^{\mu\tau} & & \\ C^T[M+e^{\mu\tau}SB]e^{\mu\tau} & [R-e^{2\mu\tau}C^TSC]e^{\mu\tau} & e^{3\mu\tau}C^TSA \\ -A^T[M+e^{\mu\tau}SB]e^{\mu\tau} & e^{3\mu\tau}A^TSC & [S-e^{2\mu\tau}A^TSA]e^{\mu\tau} \end{bmatrix} \begin{bmatrix} x_t(0) \\ \\ -e^{-\mu\tau}x_t^T(-\tau) \\ -e^{-\mu\tau}\dot{x}_t^T(-\tau) \end{bmatrix}$$

is nonnegative. We note that this form can be rewritten as

$$h_1(x_t(0), x_t(-\tau), \dot{x}_t(-\tau)) =$$

$$= e^{\mu\tau}[x_t^T(0)(M+e^{\mu\tau}B^TS), -e^{-\mu\tau}x_t^T(-\tau)C^T, -e^{-\mu\tau}\dot{x}_t^T(-\tau)A^T] \cdot$$

$$\cdot \begin{bmatrix} I + \frac{2}{\varepsilon_1} S^{-1}e^{-2\mu\tau} & I & -I \\ I & I + \frac{2}{\varepsilon_1} Se^{2\mu\tau} & Se^{2\mu\tau} \\ -I & Se^{2\mu\tau} & \varepsilon_1 Se^{2\mu\tau} \end{bmatrix} \begin{bmatrix} (M+e^{\mu\tau}SB)x_t(0) \\ -e^{-\mu\tau}Cx_t(-\tau) \\ -e^{-\mu\tau}A\dot{x}_t(-\tau) \end{bmatrix} +$$

$$+ x_t^T(0)[W - (A^T+\mu I)M - M(A+\mu I) - (R+B^TSB)e^{\mu\tau} -$$

$$- e^{\mu\tau}(M+e^{\mu\tau}B^TS)(I + \frac{2}{\varepsilon_1} S^{-1}e^{-2\mu\tau})(M+e^{\mu\tau}SB)]x_t(0) +$$

$$+ e^{-\mu\tau}x_t^T(-\tau)[R - C^T(Se^{2\mu\tau} + I + \frac{2}{\varepsilon_1} Se^{2\mu\tau})C]x_t(-\tau)$$

$$+ e^{-\mu\tau}\dot{x}_t(-\tau)[S - (1+\varepsilon_1)e^{2\mu\tau}A^TSA]\dot{x}_t(-\tau), \tag{9.2}$$

where ε_1 is an arbitrary positive number.

First of all, note that since the spectral radius of $e^{\mu\tau}A$ is strictly less than one, then for $\varepsilon_1 > 0$ sufficiently small there always exist a positive definite matrix S such that

$$S - (1+\varepsilon_1)e^{2\mu\tau}A^TSA > 0.$$

Now we choose R as

$$R = C^T(I + e^{2\mu\tau}(1 + \frac{2}{\varepsilon_1})S)C + \varepsilon_1 I;$$

R is then a positive definite matrix.

The matrix

$$W - (A^T+\mu I)M - M(A+\mu I) - (R+B^T SB)e^{\mu\tau} -$$

$$- e^{\mu\tau}(M+e^{\mu\tau}B^T S)(I + \frac{2}{\varepsilon_1} S^{-1}e^{-2\mu\tau})(M+e^{\mu\tau}SB)$$

will certainly be positive definite by taking $M = I$ and $W = k_W I$, $k_W > 0$ sufficiently large.

Finally, from the particular form of the first term in (9.2) it is easily seen that this term is positive semidefinite.

In this manner, it is seen that the form $h_1(x_t(0), x_t(-\tau), \dot{x}_t(-\tau))$ can be always made nonnegative.

All the above results can thus be summarized in the form of a

Theorem 2: Consider the matrix neutral difference-differential equation with one delay

$$\dot{x}(t) + A\dot{x}(t-\tau) = Bx(t) + Cx(t-\tau)$$

and the Liapunov functional given by equation (5.1). Let

$$\gamma = \max\{\text{Re}\ \lambda \,|\, \det[\lambda(I+Ae^{-\lambda\tau}) - B - Ce^{-\lambda\tau}] = 0\}$$

and $\varepsilon > 0$. Then there exist constant positive definite matrices M, R and S, and a differentiable matrix $P(\alpha)$, $0 \leq \alpha \leq \tau$, with $P(0) = P(0)^T$ such that the functional V is positive definite, bounded above, and

$$\dot{V} \leq 2(\gamma+\varepsilon)V.$$

Note that, if $\gamma < 0$, then the above result yields exponential asymptotic stability.

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